Int J Theor Phys (2010) 49: 98–109
DOI 10.1007/s10773-009-0182-z

Hawking-de Sitter Thermalization of Quasi-Minkowskian Massless Scalaron Production, Energy Density Content and Back-Reaction

Ana-Camelia Pîrghie · Ciprian Dariescu · Marina-Aura Dariescu

Received: 24 August 2009 / Accepted: 23 October 2009 / Published online: 3 November 2009

Abstract Projecting the closed form expression of the de Sitter scalar field operator onto the Minkowskian positive frequency massless modes, we compute the corresponding Bogolubov coefficient which is associated to the (massless) quasiparticle creation during the stationary quasi-de Sitter stage of the Universe. Thereafter, we derive the expression of the thermalized energy density which reveals an interesting mixture of de Sitter false vacuum and dark-radiation, exotic dust and black body radiation. Setting the temperature to the value of the Hawking one for the de Sitter spacetime, we finally analyze the (straightforward) back-reaction of the newly created "matter" on the scale function. It basically points out three stages of highly continuous evolution represented by an initially short radiation-like era, a somewhat long-lasting connecting phase made of coherent massless oscillations, in its beginnings, ended up by the dark-radiation (i.e. curvature-like term) contribution and, finally, a much slower exponential expansion than the initial de Sitter one.

Keywords Particle creation · de Sitter spacetime · Bogolubov coefficient · Chaotic inflation

1 Introduction

In 1917 Albert Einstein derived the first generally relativistic cosmological model as an exact solution to his famous equations for a static spatially closed homogeneously isotropic universe filled in by a smooth pressureless fluid which has come to be termed as cosmological dust. Nevertheless, in order to sustain the corresponding $S^3 \times R$ geometry, Einstein had to introduce a physically strange looking term Λ, as an universal cosmological constant which compensates the negative geometric pressure of the model. Soon, after, considering the Λ-term alone, William de Sitter discovered a seemingly static, maximally symmetric, exact solution, of constant curvature $R = 4\Lambda$, with $\Lambda > 0$, which exhibited some sort of universal red shift. In despite of Einstein's doubts with respect to the physical meaning of

A.-C. Pîrghie · C. Dariescu · M.-A. Dariescu (✉)
Faculty of Physics, Al. I. Cuza University, Bd. Carol I, no. 11, 700506 Iaşi, Romania
e-mail: marina@uaic.ro

the de Sitter solution, Arthur Eddington had shown, by his celebrated coordinate transformations, that it represents (only) the so-called stationary sector of the whole de Sitter Universe, the red shift being induced by an exponential expansion of constant argument rate $\sqrt{\Lambda/3}$. Thereafter, a new era began for modern cosmology; Hubble's observations confirmed the expansion of the universe, leading to (and also described by) his famous law, $v = Hr$, where H stands for what we call today the Hubble parameter, and the Friedmann's model emphasized the role of the matter critical energy density $3H^2/8\pi G$ with respect to the global evolution of the Universe, namely, for ever expanding if the energy density content is less or equal to the critical one, and respectively, a bouncing cyclic one for large densities. A bit latter, in the thirties, the various already formulated cosmological models have been generalized and also unified, in a consistent geometrodynamic picture based on the Cosmological Principle, by Robertson and Walker; this whole class of time-evolving homogeneous and isotropic spacetimes is termed today as the (k, Λ)-*RW* cosmologies. All of them possess at least a G_c-group of motion (G_7 for the Einstein Universe and G_{10} for the de Sitter one) and therefore the maximally symmetric spatial submanifolds can only be the three dimensional hyperboloid H^3, for $k = -1$, the three dimensional real space R^3 for $k = 0$ and the three dimensional sphere, S^3, for $k = 1$, the cosmic time (direction) being orthogonal to each of these spacelike hypersurfaces.

As we have mentioned before, one of the special Robertson-Walker exact solutions is the de Sitter spacetime. The whole of it, i.e. the covering manifold, can be easily visualized as a 4-dimensional hyperboloid embedded in a flat five-dimensional space [1], the stationary sector, i.e. the $(0, \Lambda)$-*RW* model, covering just a part of it because of the presence of the so-called de Sitter Horizon. This expanding universe had no Big-Bang in any finite past and, moreover, it respects the Perfect Cosmological Principle [2]. According to the present knowledge, our universe is probably approaching a quasi de Sitter stage in the very far future.

In the last decades, a wide interest has been focused on particle creation. In a curved spacetime the concept of particle is more subtle than in a flat spacetime, generally because there is no Lorentz symmetry, which can indicate the best vacuum state. A non-static curved spacetime may be responsible for the phenomenon of particle creation. This problem was discussed at first by E. Schrodinger [3], but an elaborate approach was made later by L. Parker [4]. In a $1 + 1$ spatially closed Robertson-Walker spacetime, this subject was investigated by N.D. Birell and P.C. Davis [5], while the particle creation for a massive scalar field conformally coupled to a spatially closed Robertson-Walker spacetime was discussed in [6], and revisited recently in [7]. Some authors have shown that the amount of particle production in an arbitrary cosmological background can be determined using only the late-time positive frequency modes alone [8]. In this respect, not only the intensive studies on generalizing the method of spacetime description of the gravitational particle creation for scalar fields with nonconformal coupling to the curvature [9], have been active topics of investigation; likewise, the particle creation in de Sitter spacetime [10] and in the conformally flat ones with a non-conformal field [11], have made important contributions to the best unification picture.

Our present interest concerns the massless quantum field fluctuations in the second mid of the chaotic inflation, discussing the so-called quasiparticle production during the stationary de Sitter stage.

2 The Linearly Independent Solutions

The de Sitter universe is described by the metric

$$ds^2 = e^{2Ht}\delta_{\mu\nu}dx^{\mu}dx^{\nu} - (dt)^2, \tag{1}$$

which can also be written as

$$ds^2 = e^{2Ht}[\delta_{\mu\nu}dx^{\mu}dx^{\nu} - e^{-2Ht}(dt)^2], \tag{2}$$

$H \in \mathbb{R}_+$ being the Hubble constant, and therefore, to enlarge the treatment with respect to the conformal time "η", it comes to be considered as

$$ds^2 = C^2(\eta)[\delta_{\mu\nu}dx^{\mu}dx^{\nu} - (d\eta)^2]. \tag{3}$$

The well-known Klein-Gordon equation for the massive real scalar field Φ

$$\frac{1}{\sqrt{|g|}}(\sqrt{|g|}g^{ik}\Phi_{,k})_{,i} - m_0^2\Phi = 0, \tag{4}$$

in conformal time, reads

$$\Delta\Phi - \frac{1}{C^2(\eta)}\frac{\partial}{\partial\eta}\left(C^2(\eta)\frac{\partial\Phi}{\partial\eta}\right) - m_0^2C^2(\eta)\Phi = 0, \tag{5}$$

where one has obviously employed

$$g_{\alpha\beta} = C^2(\eta)\delta_{\alpha\beta}, \qquad g_{44} = -C^2(\eta). \tag{6}$$

Because of the flat Laplacian $\Delta = \delta^{\mu\nu}\partial^2_{\mu\nu}$, using the variable separation

$$\Phi(\vec{x},\eta) = Ne^{i\vec{k}\cdot\vec{x}}T_{\vec{k}}(\eta), \tag{7}$$

(5) transforms into

$$\frac{d^2T_{\vec{k}}}{d\eta^2} + \frac{2}{C(\eta)}\frac{dC}{d\eta}\frac{dT_{\vec{k}}}{d\eta} + [|\vec{k}|^2 + m_0^2C^2(\eta)]T_{\vec{k}} = 0. \tag{8}$$

In order to find the de Sitter conformal scale factor $C(\eta)$ we use the connection formula between the universal time and the conformal one,

$$d\eta = e^{-Ht}dt, \tag{9}$$

leading to

$$\eta = \eta_0 - \frac{1}{H}e^{-Ht}. \tag{10}$$

In the heuristic gauge, for $H \to 0$ and $a(t) = e^{Ht} \to 1 \leftarrow C(\eta)$ i.e. $\eta \simeq t$, yielding the choice of η_0 corresponding to H^{-1}, i.e.

$$\eta = \frac{1}{H}(1 - e^{-Ht}) \in \left(-\infty, \frac{1}{H}\right), \tag{11}$$

for $t \in (-\infty, \infty)$ the conformal scale factor is

$$C(\eta) = \frac{1}{1 - H\eta}, \tag{12}$$

and the universal to conformal time relation is given by

$$t = -\frac{1}{H}\ln(1 - H\eta). \tag{13}$$

Thus, the Klein-Gordon equation for the time dependent function amplitude $T_{\vec{k}}(\eta)$ on the stationary sector of de Sitter universe, in conformal time η, reads

$$\frac{d^2 T_{\vec{k}}}{d\eta^2} + \frac{2H}{1 - H\eta}\frac{dT_{\vec{k}}}{d\eta} + \left[|\vec{k}|^2 + \frac{m_0^2}{(1 - H\eta)^2}\right]T_{\vec{k}} = 0. \tag{14}$$

Switching to the timelike dimensionless variable $\tau = 1 - H\eta \Rightarrow \tau \in (0, \infty)$, with $\tau \to \infty$ in the absolute past $\eta \to -\infty$ and $\tau \to 0$ in the absolute future $\eta \to H^{-1}$, (14) becomes

$$\frac{d^2 T_{\vec{k}}}{d\tau^2} - \frac{2}{\tau}\frac{dT_{\vec{k}}}{d\tau} + \left[\left(\frac{|\vec{k}|}{H}\right)^2 + \frac{(m_0/H)^2}{\tau^2}\right]T_{\vec{k}} = 0, \tag{15}$$

with $\tau = e^{-Ht}$ as the direct time conversion relation. Using the notations

$$\kappa = \frac{|\vec{k}|}{H}, \qquad \mu = \frac{m_0}{H}, \qquad s = \kappa\tau, \tag{16}$$

(as the new physically dimensionless and positive definite timelike variable $s \in \mathbb{R}_+$), it almost takes the form of a Bessel equation,

$$\frac{d^2 T_{\vec{k}}}{ds^2} - \frac{2}{s}\frac{dT_{\vec{k}}}{ds} + \left[1 + \frac{\mu^2}{s^2}\right]T_{\vec{k}} = 0, \tag{17}$$

where explicitly $s = \frac{|\vec{k}|}{H}e^{-Ht}$ on $\mathbb{R}_+$ as the direct time conversion relation. In order to convert it properly to some Bessel equation [12]

$$\frac{d^2 Z_\nu}{ds^2} + \frac{1}{s}\frac{dZ_\nu}{ds} + \left[1 - \frac{\nu^2}{s^2}\right]Z_\nu = 0, \tag{18}$$

one needs the function substitution,

$$T_{\vec{k}} = s^\alpha Z_\nu(s), \tag{19}$$

so that, it yields (replacing it into the $T_{\vec{k}}$-equation),

$$\frac{d^2 Z_\nu}{ds^2} + \frac{2\alpha}{s}\frac{dZ_\nu}{ds} + \frac{\alpha(\alpha-1)}{s^2}Z_\nu - \frac{2}{s}\left(\frac{dZ_\nu}{ds} + \frac{\alpha}{s}Z_\nu\right) + \left(1 + \frac{\mu^2}{s^2}\right)Z_\nu = 0, \tag{20}$$

i.e.

$$\frac{d^2 Z_\nu}{ds^2} + \frac{2(\alpha-1)}{s}\frac{dZ_\nu}{ds} + \left(1 - \frac{\alpha(3-\alpha) - \mu^2}{s^2}\right)Z_\nu = 0. \tag{21}$$

Thence, one should impose $2(\alpha-1)=1$, meaning $\alpha=3/2$ and $\nu^2=\frac{9}{4}-\mu^2$, to get precisely the Bessel equation (18), with $\nu=[\frac{9}{4}-(\frac{m_0}{H})^2]^{1/2}$ explicitly. Subsequently, in the complex representation, the two linearly independent solutions for the time-dependent amplitude function can be written as

$$T_{\vec{k}}=s^{3/2}\{H_\nu^{(1)}(s),\,H_\nu^{(2)}(s)\},\tag{22}$$

where $H_\nu^{(1,2)}$ are the Hankel function, which for the particular case of a massless field, $m_0=0$, do readily become [13]

$$H_{3/2}^{(1)}=-\sqrt{\frac{2}{\pi s}}e^{is}\left(1+\frac{i}{s}\right),\qquad H_{3/2}^{(2)}=-\sqrt{\frac{2}{\pi s}}e^{-is}\left(1-\frac{i}{s}\right).\tag{23}$$

3 The Scalar Field Operator

Going back to the Gordon equation for the massive real scalar field (4), we write down the full solution as an orthonormal modes superposition

$$\Phi(x)=\int d^3p[c(\vec{p})u_{\vec{p}}(x)+c^+(\vec{p})\bar{u}_{\vec{p}}(x)],\tag{24}$$

where $c(\vec{p})$ is the annihilation operator for a particle of momentum $\vec{p}$, $c^+(\vec{p})$ is the creation operator for a particle of momentum $\vec{p}$, $u_{\vec{p}}(x)$ are the orthonormal positive frequency mode, $\bar{u}_{\vec{p}}(x)$ are the orthonormal negative frequency mode.

Thus, the scalar field in the de Sitter and Minkowski space can be respectively written as

$$\begin{aligned}\Phi_{dS}(x)&=\int[c_{\vec{p}}u_{\vec{p}}(x)+c^+_{\vec{p}}\bar{u}_{\vec{p}}(x)]d^3p,\\ \Phi_M(x)&=\int[a_{\vec{p}}v_{\vec{p}}(x)+a^+_{\vec{p}}\bar{v}_{\vec{p}}(x)]d^3p,\end{aligned}\tag{25}$$

with the Bugolubov linear relations between the two sets $\{a,a^+\}$, $\{c,c^+\}$, of annihilation-creation operators

$$\begin{aligned}a_{\vec{p}'}&=\int\alpha_{\vec{p}'\vec{p}}c_{\vec{p}}d^3p+\int\beta_{\vec{p}'\vec{p}}c^+_{\vec{p}}d^3p,\\ \alpha_{\vec{p}'\vec{p}}&=(v_{\vec{p}'},u_{\vec{p}}),\qquad \beta_{\vec{p}'\vec{p}}=(v_{\vec{p}'},\bar{u}_{\vec{p}}),\end{aligned}\tag{26}$$

where $v_{\vec{p}}(x)=\frac{e^{ipx}}{\sqrt{(2\pi)^3 2E_{\vec{p}}}}$ stands for the completely ortho-normal set of positive-frequency modes in Minkowski spacetime.

The coefficients $\beta_{\vec{p}'\vec{p}}$ are related by their square modulus to the number of particles created by the gravitational field. In the Hilbert space, spanned by the quantum states the orthonormal frequency modes are satisfying the condition

$$(u_{\vec{p}'},u_{\vec{p}})=\delta(\vec{p}'-\vec{p}),\tag{27}$$

where the relativistic invariant scalar product is defined as

$$(u_{\vec{p}'},u_{\vec{p}})\stackrel{\Delta}{=}i\int(\bar{u}_{\vec{p}'}u_{\vec{p},t}-\bar{u}_{\vec{p}',t}u_{\vec{p}})\sqrt{-g}d^3x.\tag{28}$$

The respective positive and negative frequency modes are expressed by

$$u_{\vec{p}}(x) = N_{\vec{p}} e^{-\frac{3}{2}Ht} H_\nu^{(1)}\left(\frac{|\vec{p}|}{H} e^{-Ht}\right) e^{i\vec{p}\cdot\vec{x}},$$
$$\bar{u}_{\vec{p}}(x) = N_{\vec{p}} e^{-\frac{3}{2}Ht} H_\nu^{(2)}\left(\frac{|\vec{p}|}{H} e^{-Ht}\right) e^{-i\vec{p}\cdot\vec{x}}, \tag{29}$$

where the normalization coefficient reads

$$N_{\vec{p}} = \frac{(4\pi)^{-1}}{\sqrt{2H}}. \tag{30}$$

4 The Massless Quasiparticles Creation and the Thermalized Energy Density Structure

For ultrarelativistic particles as well as for the massless ones $m_0 = 0$, the Hankel function does asymptotically become

$$H_{3/2}^{(1)}\left(\frac{|\vec{p}|}{H} e^{-Ht}\right) \cong \sqrt{\frac{2H}{\pi p}} e^{\frac{H}{2}t} \cdot \exp\left[i\left(\frac{p}{H}e^{-Ht} - \frac{3\pi}{4} - \frac{\pi}{4}\right)\right], \tag{31}$$

thereafter the normalization coefficient converts into

$$N_{\vec{p}} = -\frac{(4\pi)^{-1}}{\sqrt{2H}} e^{-i\frac{p}{H}}, \tag{32}$$

and therefore the orthonormal positive frequency modes in de Sitter universe are given by

$$u_{\vec{p}}(x) = -\frac{(4\pi)^{-1}}{\sqrt{2H}} e^{-\frac{3}{2}Ht} H_{3/2}^{(1)}\left(\frac{|\vec{p}|}{H} e^{-Ht}\right) \exp\left[i\left(\vec{p}\cdot\vec{x} - \frac{|\vec{p}|}{H}\right)\right]. \tag{33}$$

As we know, the pair creation amplitude, in non-stationary gravitational fields, is given by the Bogolubov coefficient

$$\beta_{\vec{p}'\vec{p}}(t) = (v_{\vec{p}'}, \bar{u}_{\vec{p}}) = i\int (\bar{v}_{\vec{p}'}\bar{u}_{\vec{p},4} - \bar{v}_{\vec{p}',4}\bar{u}_{\vec{p}}) d^3x, \tag{34}$$

and the corresponding calculations, using (34), lead to

$$\beta_{\vec{p}'\vec{p}}(t) = -i\frac{(4\pi)^{-1}(2\pi)^3\delta(\vec{p}'+\vec{p})}{(32\pi^3 H|\vec{p}'|)^{1/2}} \exp\left[i\left(|\vec{p}'|t + \frac{|\vec{p}|}{H}\right)\right]\cdot e^{-\frac{3}{2}Ht}$$
$$\times \left\{\frac{dH_\nu^{(2)}}{dt} - \left(\frac{3}{2}H + i|\vec{p}'|\right)H_\nu^{(2)}\right\}, \tag{35}$$

respectively,

$$\beta_{\vec{p}'\vec{p}}(t) = -i\frac{(4\pi)^{-1}(2\pi)^3\delta(\vec{p}'+\vec{p})}{(32\pi^3 H|\vec{p}'|)^{1/2}} \exp\left[i\left(|\vec{p}'|t + \frac{|\vec{p}|}{H}\right)\right]\cdot\sqrt{\frac{2H}{\pi|\vec{p}|}}$$
$$\times e^{-Ht}\exp\left(-i\frac{|\vec{p}|}{H}e^{-Ht}\right)\left(i|\vec{p}'| - i|\vec{p}|e^{-Ht} + \frac{|\vec{p}'|}{|\vec{p}|}He^{Ht}\right). \tag{36}$$

Consequently, the number density of particles with momentum $|\vec{p}|$, defined as (and numerically given by)

$$f_p(t) = V^{-1}\left|\beta_{\vec{p}'\vec{p}}(t)\right|^2,$$

has the following form

$$f_p(t) = \frac{1}{(2\pi)^3 \cdot 4|\vec{p}|^2}[H^2 + |\vec{p}|^2(1-e^{-Ht})^2 e^{-2Ht}], \tag{37}$$

and therefore the thermalized particle density and the corresponding energy density can be expressed by the convolutions

$$\begin{aligned} n_\beta(t) &= \frac{1}{8\pi^2}\int_0^\infty [H^2 + p^2(1-e^{-Ht})^2 e^{-2Ht}] f_\beta(p)dp, \\ w_\beta(t) &= \frac{1}{8\pi^2}\int_0^\infty p[H^2 + p^2(1-e^{-Ht})^2 e^{-2Ht}] f_\beta(p)dp, \end{aligned} \tag{38}$$

where we have denoted by $f_\beta(p) = (e^{\beta p} - 1)^{-1}$, the Bose-Einstein distribution function, with $\beta = 1/k_B T$.

As it can be noticed, the energy density at the initial moment, $t = 0$, reads

$$w_\beta(t) = \frac{1}{8\pi^2}\int_0^\infty \frac{pH^2}{e^{\beta p} - 1}dp, \tag{39}$$

and it points out that no particle creation would occur if the Hubble parameter were vanishing, ($H = 0$), fulfilling therefore the Heuristic Principle based on the coincidence of the two vacua at $t = 0$ and on the flatness of de Sitter metric at $H = 0$.

Thence, coming back to the full energy density expression derived in (38), that one becomes

$$w_\beta(t) = \frac{H^2}{8\pi^2}\int_0^\infty \frac{p}{e^{\beta p} - 1}dp + \frac{e^{-2Ht}}{8\pi^2}(1-e^{-Ht})^2\int_0^\infty \frac{p^3}{e^{\beta p} - 1}dp, \tag{40}$$

and performing the momenta integrals, it does finally read

$$w_\beta(t) = \frac{H^2}{48\beta^2} + \frac{\pi^2}{120\beta^4}\left(\frac{1}{a^2} - \frac{2}{a^3} + \frac{1}{a^4}\right), \tag{41}$$

where we have identified

$$e^{Ht} = a,$$

in order to increase a bit its generality. As it can be noticed, the first term is alike a cosmological one, while the ones in the parenthesis do respectively correspond to dark-radiation (or, curvature term), "exotic" dust and thermalized (black body) radiation.

5 The Straightforward Back Reaction

According to the previous results, the full Einstein equations become

$$
\begin{aligned}
&2\frac{d^2 f}{dt^2} + 3\left(\frac{df}{dt}\right)^2 + \frac{k}{a^2} - \Lambda = -\kappa_0 P, \\
&3\left(\frac{df}{dt}\right)^2 + 3\frac{k}{a^2} - \Lambda = \kappa_0 w_\beta(t),
\end{aligned}
\tag{42}
$$

where the pressure term, can be derived from the well-known energy conservation equation

$$
\frac{dw_\beta}{dt} + 3\frac{\dot{a}}{a}(w_\beta(t) + P) = 0, \tag{43}
$$

the dot standing for time derivative. Thus, one finds for the pressure the following expression

$$
P = -\frac{H^2}{48\beta^2} - \frac{\pi^2/360}{\beta^4 a^2} + \frac{\pi^2/360}{\beta^4 a^4}, \tag{44}
$$

considering the factor $\beta = \frac{1}{k_B T}$ as being time independent.

Using the second equation of the system (42), i.e. the Friedmann's one, and the energy density expression (41), one readily gets the essential differential equation

$$
\frac{df}{dt} = \pm\left\{\frac{\Lambda}{3} + \frac{\kappa_0}{3}\left[\frac{H^2}{48\beta^2} + \frac{\pi^2}{120\beta^4}\left(\frac{1}{a^2} - \frac{2}{a^3} + \frac{1}{a^4}\right)\right] - \frac{k}{a^2}\right\}^{1/2}. \tag{45}
$$

The analogy between a black hole interior and the de Sitter space [13] leads to the Hawking temperature $T_H = \frac{H}{2\pi}$ so that

$$
\beta = \frac{2\pi}{H}, \tag{46}
$$

in "natural units" $k_B = 1 = h/2\pi$. For the case of a flat three dimensional space with no cosmological constant ($k = 0 = \Lambda$), taking into account the previous relation (46), it yields for (45) the intermediary form,

$$
\frac{df}{dt} = \frac{H^2\sqrt{\kappa_0}}{24\pi}\left[1 + \frac{1}{10}\left(\frac{1}{a^2} - \frac{2}{a^3} + \frac{1}{a^4}\right)\right]^{1/2}. \tag{47}
$$

Employing the relation between the primitive of the Hubble function f and the scale factor a, namely

$$
f = \ln a \quad \Leftrightarrow \quad a = e^f,
$$

the expression (47) becomes

$$
\frac{da}{dt} = \frac{H^2\sqrt{\kappa_0}}{24\pi}\left[a^2 + \frac{1}{10}\left(1 - \frac{1}{a}\right)^2\right]^{1/2}, \tag{48}
$$

and if we choose the "absolute" Planck units $\kappa_0 = \frac{8\pi}{M_P^2}$, we notice that the above equation can be written as

$$
\frac{da}{dt} = \frac{H^2/M_P}{6\sqrt{2\pi}}\left[a^2 + \frac{1}{10}\left(1 - \frac{1}{a}\right)^2\right]^{1/2}. \tag{49}
$$

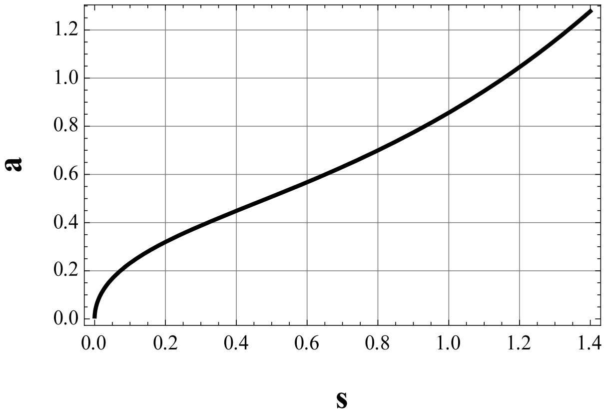

Fig. 1 The evolution of the scale function a, with respect to the rescaled cosmic time s, yielding from (49), with the initial condition $a_0 = 10^{-2}$

For the asymptotic behaviour of the scale function, it follows that, for $0 < a \ll 1$, one gets the dependence

$$a(t) = (6\sqrt{5\pi})^{-1/2}\sqrt{\underline{t}}, \quad \underline{t} = \frac{H^2}{M_P}t, \tag{50}$$

which does typically stand for a radiation-dominated stage. The corresponding solution for large values of a, i.e. $a \gg 1$, will be accordingly given by

$$a = a_* \exp\left(\frac{\sqrt{2\pi}}{12\pi}\underline{t}\right), \tag{51}$$

being clearly related to an accelerated expansion of the universe.

A numerical integration of the full first-order differential equation (49), written as

$$\frac{da}{ds} = \left[a^2 + \frac{1}{10}\left(1 - \frac{1}{a}\right)^2\right]^{1/2},$$

with respect to the rescaled, physically dimensionless, (cosmic) time parameter

$$s = \frac{H^2/M_P}{6\sqrt{2\pi}}t$$

and with some initial condition, $0 < a(s = 0) = a_0 \ll 1$, say $a_0 = 10^{-2}$, confirms the previously mentioned asymptotic behaviour(s) and, in addition, it clearly reveals a "long lasting" connecting phase made of a mixture of coherent massless excitations and the curvaturelike term contribution, as it can be noticed in the Fig. 1.

6 Summary and Discussions

As it turns out from the basic equations of the chaotic inflationary scenario,

$$\frac{dH}{dt} = -\frac{4\pi}{M_P^2}\left(\frac{d\Phi}{dt}\right)^2,$$

$$\frac{d^2\Phi}{dt^2} + 3H\frac{d\Phi}{dt} + m_*^2\Phi = 0,$$

$$H^2 = \frac{4\pi}{3M_P^2}\left[\left(\frac{d\Phi}{dt}\right)^2 + m_*^2\Phi^2\right],$$

where $H = \frac{df}{dt}$ and m_* is the mass of the coherent inflaton field Φ, M_P standing for the Planck mass, there is a special regime—which goes by the name "uniform slow rolling" of the inflaton—where both the Hubble function H and the scalar field Φ are linearly decreasing according to the laws

$$H = H_0\left[1 - \frac{m_*^2}{3H_0}t\right],$$

$$\Phi = \Phi_0\left[1 - \frac{m_*^2}{3H_0}t\right].$$

The two initial values $\{H_0, \Phi_0\}$ are not independent, being related by the "curvature" equation,

$$\frac{dH}{dt} = -\frac{4\pi}{M_P^2}\left(\frac{d\Phi}{dt}\right)^2,$$

namely

$$\frac{4\pi}{3H_0^2}\left(\frac{m_*}{M_P}\right)^2\Phi_0^2 = 1,$$

yielding the "characteristic" value of the Hubble constant with respect to the initial value of the inflaton, i.e.

$$H_0 = \sqrt{\frac{4\pi}{3}}\frac{m_*}{M_P}\Phi_0.$$

As it can be noticed from the Friedmann equation

$$H^2 = \frac{4\pi}{3M_P^2}\left[\left(\frac{d\Phi}{dt}\right)^2 + m_*^2\Phi^2\right],$$

so long as

$$0 \le \left(\frac{d\Phi}{dt}\right)^2 \ll m_*^2\Phi^2,$$

meaning "slow motion" indeed, on the (positive) time-interval $\Delta t \ll \frac{m_*^2}{3H_0}$, the Hubble function does basically stand around the constant value H_0, leading therefore to the quasi-de Sitter stage,

$$f = Ht \quad \text{i.e. } a(t) \stackrel{\Delta}{=} e^{f(t)} = e^{Ht},$$

which has been considered in the present paper. Next, we have focussed on the massless (or ultrarelativistic) excitations $\Phi(\vec{x}, t)$ of the inflaton deriving the ortho-normal set of positive-frequency modes (33), which, inserted into (24), lead to the canonically quantized form of the de Sitter (scalar) field operator $\Phi_{dS}(\vec{x}, t)$ mentioned in (25). According to the exact form

(23) of the Hankel function $H^{(1)}_{3/2}$, for the positive-frequency massless modes, it yields from (33), the corresponding mode

$$u_{\vec{p}}\,(\vec{x},t)=\frac{e^{i\,\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3\,2\,|\vec{p}|}}\exp\left[i\left(\frac{|\vec{p}|}{H}e^{-Ht}-\frac{|\vec{p}|}{H}\right)\right]e^{-Ht}\left(1+\frac{iH}{|\vec{p}|}e^{Ht}\right).$$

As it can be noticed, in view of the Heuristic Principle, if one takes the limit $H\to 0_+$, the above expression becomes

$$u_{\vec{p}}\,(\vec{x},t)\,|_{H\to 0_+}=\frac{e^{i\,\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3\,2\,|\vec{p}|}}\exp\left[i\left(\frac{|\vec{p}|}{H}-\frac{|\vec{p}|}{H}-|\vec{p}|\,t\right)\right],$$

i.e. precisely the free massless mode on Minkowski spacetime

$$v_{\vec{p}}\,(\vec{x},t)=\frac{e^{i\,(\vec{p}\cdot\vec{x}-|\vec{p}t|)}}{\sqrt{(2\pi)^3\,2\,|\vec{p}|}}.$$

One may also notice that, at $t=0$, the de Sitter mode,

$$u_{\vec{p}}\,(\vec{x},t)=\frac{e^{i\,\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3\,2\,|\vec{p}|}}\left(1+\frac{iH}{|\vec{p}|}\right),$$

comes as close as possible to the Minkowski one,

$$v_{\vec{p}}\,(\vec{x},0)=\frac{e^{i\,\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3\,2\,|\vec{p}|}},$$

the supplementary term being (intuitively) related to the scattering of the massless Minkowski mode onto the constant positive curvature of the de Sitter spacetime, $R=12H^2$. That is why we have projected the whole de Sitter operator onto the "corresponding" positive-frequency Minkowski massless modes, being particularly interested in the missing of $\{a,c^+\}$-operators given by the Bogolubov coefficient $\beta_{\vec{p}'\vec{p}}(t)$ expressed in full as in (36). Next, by tacitly integrating its square modulus over the momentum space, we have derived the created quasi-particles number density distribution function $f_{\vec{p}}(t)$—given by (37)—in a state of momentum $\vec{p}$ at a cosmic time value t. Being bosons, we have "thermalized" them according to the Bose-Einstein distribution function

$$f_\beta\,(|\vec{p}|)=\left[e^{\beta|\vec{p}|}-1\right]^{-1}$$

and have gotten the produced matter-content, during the quasi-de Sitter stage, as revealed by the energy density terms mentioned in (41). As it can be noticed, they do respectively correspond to a slight positive "correction" of the cosmological term Λ, which might explain the post-inflationary accelerated expansion of the (observed) universe, a positive curvature term, which might indicate that we live after all in a topologically closed universe, some "exotic dust" related to the missing baryonic-matter problem and "last but not least", a thermalized radiation term which might be necessary for the reheating process (after the chaotic inflationary stage).

Finally, just for play, because of the presence of the de Sitter event horizon $d_H=H^{-1}$, following the analogy to the interior of the corresponding black hole, we have chosen the temperature β^{-1} as being the Hawking one, $T_H = H/(2\pi)$, and watched out the back reaction described by the first-order differential equation (49). It primarily leads to a radiation dominated era, followed by a mixture with the coherent excitations of the massless field and ends up in an accelerated expansion stage of the back reacted universe.

References

1. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Spacetime. Cambridge University Press, Cambridge (1973)
2. Robertson, H.P., Noonan, T.W.: Relativity and Cosmology. Sauders, Philadelphia, (1968)
3. Schrodinger, E.: The proper vibrations of the expanding universe. Physica **6**, 899–912 (1939)
4. Parker, L.: Particle creation in expanding universes. Phys. Rev. Lett. **21**, 562–564 (1968)
5. Birrel, N.D., Davies, P.C.: Quantum Fields in Curved Space. Cambridge University Press, Cambridge (1982)
6. Setare, M.R.: Cosmological particle creation and dynamical Casimir effect. Int. J. Theor. Phys. **43**, 2237–2242 (2004)
7. Pascoal, F., Farina, C.: Particle creation in a Robertson-Walker Universe revisited. Int. J. Theor. Phys. **46**, 2950–2955 (2007)
8. Hamilton, A., Kabat, D., Parikh, M.: A first-quantized formalism for cosmological particle production. J. High Energy Phys. **0407**, 024 (2004)
9. Pavlov, Yu.V.: Spacetime description of scalar particle creation by a homogenous isotropic gravitational field. Gravit. Cosmol. **14**, 314–320 (2008)
10. Zhao, W.: Particle creation in de sitter spacetime and time quantization. arXiv:gr-qc/0505148 (2005)
11. Molina-Paris, C.: Energy-momentum tensor and particle creation in the de sitter universe. In: Piran, T. (ed.) Proceedings of the Eight Marcel Grossmann Meeting, on Recent Developments in Theoretical and Experimental General Relativity, Gravitation, and Relativistic Field Theories, pp. 827–829. World Scientific, Singapore (1997)
12. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 4th edn. Academic Press, New York (1965)
13. Linde, A.: Particle Physics and Inflationary Cosmology. Harwood Academic, Harwood (1990)